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LETTER TO THE EDITOR

Multifractality and the Kauffman model

Naeem Jan

Physics Department, St Francis Xavier University, Antigonish, Nova Scotia, Canada B2G 1C0

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Abstract. We simulate the spread of an initial damage in the Kauffman cellular automata at its critical point on the square lattice. We calculate various moments of the probability that a site has been damaged n times and check for multifractality in the critical exponents for different ensembles. Specifically we find no evidence for multifractality when the moments of the probabilities are evaluated with lattice size, L, but multifractal behaviour occurs when the moments are monitored as a function of time, t.

Multifractality [1-4] has been recently observed in many different systems. The distribution of voltages in random resistor networks, or the distribution of growth probabilities in a diffusion-limited aggregate, are two typical examples of the occurrence of multifractal behaviour. The various moments, M_q, of these distributions each scale with a unique exponent which depends on q and the exponents are not simply related to each other as is the case, for example, with the various moments of the distribution of percolating clusters [5]. We would like to elaborate on the recent report of multifractal behaviour in the Kauffman model [6].

The Kauffman [7] model is a deterministic cellular automata that was introduced to investigate the effects of the switching properties of the gene. The state of a central spin at time t is only determined by the states of its neighbours at time $(t-1)$. At the start of a trial 16 rules are assigned to each site: for each of the 16 states of the neighbours a random number r is compared to p_c ($\sim$0.29 for the square lattice)†, and the central spin will be assigned the state up if r is less than p_c, otherwise it will be placed in the state down. These rules persist for the duration of the trial and for all subsequent times the central spin will flip at the next time step to the state designated by its neighbours. The system is allowed to relax from an initial random state to 'equilibrium'. A lattice is cloned, i.e. the spins and corresponding rules are identical for the replica, with one exception, in that the central spin is placed in the opposite state. This is the initial damage. The two lattices are allowed to evolve and the damage monitored, i.e. the cloud of sites [8] which is the number of sites in different spin states between the two lattices. This is also called the 'overlap' or Hamming distance.

We tabulate for each site i, the number of times, n_i, a given site is damaged as a function of time and for each i calculate the probability of being damaged $p_i - n_i/\Sigma\, n_i$. At least 100 trials were performed for each data point.

† It is now believed that p_c may be somewhat higher (0.295-0.298) but our main conclusions are insensitive to the exact value of p_c.

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We searched for multifractality in several ensembles as follows. (i) We calculate the moments $M_q = \Sigma\, p_i^q$ for only those damage clouds that touch the edge of the lattice of size L and only at that instant of time. We considered lattices in the range $20 \leqslant L \leqslant 250$. $M_q = 1$ for $q = 1$. For large L we expect

$$M_q \sim L^{\phi(q)}.$$

Since the number of damaged sites $M_0 \sim L^{d_f}$ we have $\Sigma_i\, n_i \sim L^{d_f+1}$. Therefore the moments are expected to scale as $M_q = \Sigma\,(n_i/\Sigma_i\, n_i)^q \sim L^{d_f(1-q)}$.

If there is no multifractal behaviour then $\phi(q) = d_f(1-q)$. In figure 1 the open circles show $\phi(q)$ against q at $p = 0.29$. We find no evidence for multifractal behaviour and $d_f = 1.67 \pm 0.03$ in good agreement with previous results.

(ii) We next consider the ensemble for all damage clouds that persist after at least 2^8 Monte Carlo time steps. This set includes those clusters that will eventually propagate and touch the edge of the infinite lattice and also those clouds which are finite in extent but where the damage persists by cycling within a local cluster of sites. The moments scale as

$$M_q \sim t^{\phi(q)}$$

and in figure 1 (open triangles) we note two distinct regions; for $q < 1$ $\phi(q)$ seems to vary continuously whilst for $q \geqslant 1$, $\phi(q) = 0$. This behaviour can be understood in that the moments are dominated by those sites that are cyclically damaged.

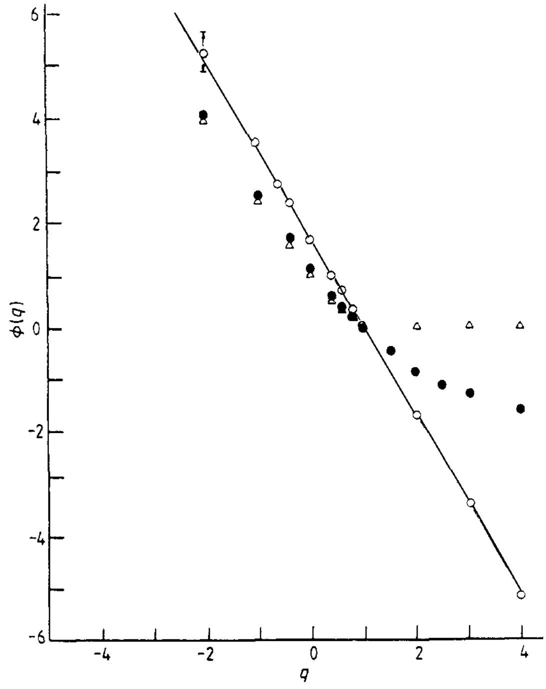

Figure 1. The critical exponents $\phi(q)$ as a function of q for: (○) case (i) for only those damage clouds that touch the edge of the lattice at size L and only at that instant of time; (△) case (ii) for all clusters that persist after 2^8 time steps; (●) case (iii) for only those clouds that propagate to the edge of the lattice. For case (i) $M_q = \Sigma\, p_i^q \sim L^{\phi(q)}$, the slope is 1.69 and $\phi(0) = 1.66$. For cases (ii) and (iii) $M_q = \Sigma\, p_i^q \sim t^{\phi(q)}$, see text for discussion.

(iii) We finally consider only those damage clusters that touch the edge of the lattice for $L = 100$ and 150. For these clusters we calculate M_q as a function of time. The data points for $L = 100$ and 150 are indistinguishable and here (figure 1, full circles) we see clear evidence for multifractality confirming the earlier results of Coniglio *et al* [6].

To understand the results shown in figure 1 we have plotted in figure 2 the frequency of first passage time against ln (time) for $L = 100$. This curve is very similar to the voltage distribution curve observed for the random resistor network [1] and the growth probabilities of diffusion-limited aggregation. These distributions all share the feature that they have a pronounced peak with a long tail. This broad distribution and the absence of a unique characteristic time give rise to multifractality when time is used as the variable. Similar differences between various moments as a function of time and as a function of distance were also observed for diffusion [9].

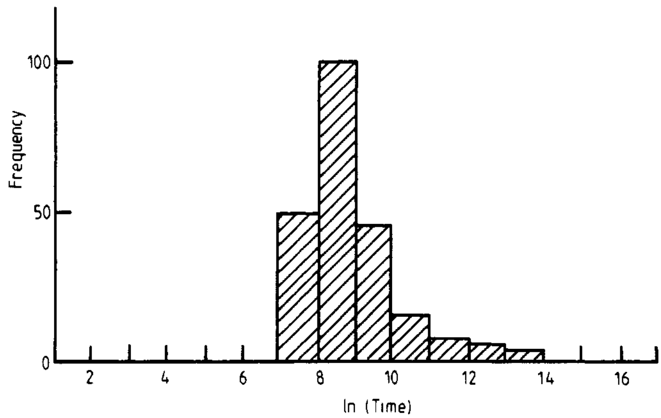

Figure 2. The frequency of first passage time plotted against ln (time) for lattice size $L = 100$.

We have simulated various ensembles of the Kauffman model and checked for multifractal behaviour. No evidence for multifractal behaviour is found when the various moments of the probabilities are plotted as a function of the average spanning time or equivalently as a function of lattice size L. However we do find multifractal behaviour when the various moments of the probability distribution for those damage clouds that eventually span the lattice are plotted as a function of time.

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